THE

MATHEMATICAL GAZETTE.

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AREAL COORDINATES.

SECTION III.

THE GENERAL EQUATION OF THE SECOND DEGREE (*continued*).

Let x_0, y_0, z_0 be the areal coordinates of a particular point, x, y, z the areal coordinates of any other point. Put $\xi=x-x_0$, etc. Then $\Sigma\xi=0$, and ξ, η, ζ (or any two of them) serve to determine the position of a point, and a relation between them gives a locus.

1. $\phi=0$ becomes $\phi(\xi, \eta, \zeta)+\phi(x_0, y_0, z_0)+\Sigma\xi\phi_{x_0}=0$.

If x_0, y_0, z_0 be the centre,

$$\phi_{x_0}=\phi_{y_0}=\phi_{z_0}, \text{ and } \phi(x_0, y_0, z_0)=-\frac{H}{K};$$

$$\therefore \phi(x, y, z)=0 \text{ becomes } K\phi(\xi, \eta, \zeta)=H.$$

Again, $\Sigma\xi=0$,

$$\therefore \phi(\xi, \eta, \zeta)=-\Sigma\eta\zeta(v+w-2u')=\Sigma\eta\zeta\frac{dK}{du},$$

or

$$\phi(\xi, \eta, \zeta)=\Sigma\xi^2(u+u'-v'-w')=\tfrac{1}{2}\Sigma\xi^2\frac{dK}{du'}.$$

Eliminating ζ between $\phi(\xi, \eta, \zeta)$ and $\Sigma\xi=0$, and we have a quadratic for ξ/η. Hence $\phi(\xi, \eta, \zeta)=0$ represents two straight lines—the asymptotes of $K\phi(\xi, \eta, \zeta)=H$, or of all homothetic concentric conics which are included in

$$\phi(\xi, \eta, \zeta)=C.$$

[NOTE. $\Sigma ux^2+2\Sigma u'yz=0$ where $\Sigma x=0$ can only be written in *one* way, so as to consist entirely of product terms, and only in *one* way without product terms.]

2. *Conditions that general equation may represent a circle.*

The equation of a circle, radius ρ, centre (x_0, y_0, z_0), is clearly $-\rho^2=\Sigma a^2\eta\zeta$. Comparing with $H=K\Sigma\frac{dK}{du}\eta\zeta$, we have the condition and the radius:

$$\frac{1}{a^2}\frac{dK}{du}=\ldots=\ldots=-\frac{1}{\rho^2}\frac{H}{K}$$

3. *Axes of General Conic.*

The distance between the centre and (x, y, z) is given by $-\rho^2=\Sigma a^2\eta\zeta$. For the axes we must find the max. and min. values of the R.H.

We have $$\Sigma(c^2\eta+b^2\zeta)\,d\xi=0\,;$$

subject to $$\Sigma\left(\frac{dK}{dw}\eta+\frac{dK}{dv}\zeta\right)d\xi\,;\ \text{and}\ \Sigma d\xi=0.$$

Eliminating $d\xi$, $d\eta$, $d\zeta$, we get, after some reduction, the equation to the axes,

$$\Sigma\frac{\xi^2}{a^2}\left(\frac{1}{b^2}\frac{dK}{dv}-\frac{1}{c^2}\frac{dK}{dw}\right)=0.$$

For the lengths of axes we use undetermined multipliers. Choose P and Q, so that

$$\left(\eta\frac{dK}{dw}+\zeta\frac{dK}{dv}\right)+P(c^2\eta+b^2\zeta)+Q=0,$$

with two similar expressions. These give $2H/K-2P\rho^2=0\,;$

$$\therefore\ \eta\left(\frac{dK}{dw}+\frac{c^2H}{\rho^2K}\right)+\zeta\left(\frac{dK}{dv}+\frac{b^2\,H}{\rho^2\ K}\right)+Q=0,\ \text{etc.},$$

and we also have $\Sigma\xi=0$.

Thus we get, on eliminating and reducing,

$$K^3\rho^4-HK\rho^2\Sigma\frac{dK}{du}bc\,.\cos A-4H^2\Delta^2=0.$$

Cor. Transformation leaves the lengths of the axes unaltered, therefore the ratios $K^3 : HK\Sigma\dfrac{dK}{du}bc\,.\cos A : 4H^2\Delta^2$ are invariants.

4. The general conic is therefore seen from (3) to be an Ellipse if K is negative, an Hyperbola if K is positive, a Parabola if K is zero, a Rectangular Hyperbola if K is positive and $\Sigma\dfrac{dK}{du}bc\cos A=0$.

The equation of the axes of the ellipse $\Sigma\sqrt{x}=0$ is $\Sigma(b^2-c^2)(y-z)^2=0$ and the lengths are given by the equation $54\rho^4-3\rho^2(a^2+b^2+c^2)+2\Delta^2=0$.

[*Pemb., Gonv. and Caius*, 1897.]

These results can be at once obtained by the method, or deduced from the results, of Art. 4.

5. *The Director Circle.*

The tangents from (x, y, z) being at right angles, the locus of (x, y, z) is

$$\Sigma x^2(b^2W+c^2V+2bcU'\cos A)$$
$$-2\Sigma yz(a^2U'-bcU\cos A+caW'\cos B+abV'\cos C)=0.$$

This may also be written

$$\Sigma x\,\Sigma x(b^2W+c^2V+2bcU'\cos A)+K\Sigma a^2yz=0.$$

6. *Condition that two lines may be conjugate.*

If f, g, h is the pole of $\Sigma lx=0$, then (§ II. 3)

$$\frac{uf+w'g+v'h}{l}=\ldots=\ldots,\ \text{or}\ \frac{f}{lU+mW'+nV'}=\ldots=\ldots.$$

Hence, if $\Sigma lx=0$ and $\Sigma l'x=0$ are conjugates,

$$\Sigma ll'U+\Sigma U'(mn'+m'n)=0.$$

The condition that the triangle formed by the lines $\Sigma l_1 x=0$, etc., may be self-polar with respect to any conic, with regard to which the triangle of reference is self-polar, is easily found.

The intersection of $\Sigma l_2 x=0$ and $\Sigma l_3 x=0$ is

$$\frac{x}{L_1}=\frac{y}{M_1}=\frac{z}{N_1}$$

where $L_1 \equiv m_2 n_3 - m_3 n_2$.

The polar of this with respect to $\Sigma u x^2=0$ is $\Sigma l_1 x=0$;

$$\therefore \frac{uL_1}{l_1}=\frac{vM_1}{m_1}=\frac{wN_1}{n_1}=\lambda_1 \text{ (say)};$$

$$\therefore uL_1=l_1\lambda_1, \text{ etc., and similarly } uL_2=l_2\lambda_2, \text{ etc.};$$

$$\therefore \lambda_1 l_1 m_1+\lambda_2 l_2 m_2+\lambda_3 l_3 m_3=0, \text{ with two similar equations};$$

$$\therefore \begin{vmatrix} 1/l_1, & 1/l_2, & 1/l_3 \\ . & . & . \\ . & . & . \end{vmatrix}=0.$$

$$\therefore \text{ etc.}$$

[*Magd. (Cam.)*, 1900.]

7. *The Foci.*

Any two lines through (x_0, y_0, z_0) have as their equations

$$\Sigma lx=\lambda, \ \Sigma l'x=\lambda', \text{ where } \lambda \equiv \Sigma l x_0, \ \lambda'=\Sigma l' x_0.$$

Applying the conditions of (6) and reducing, these lines are conjugate if

$$\Sigma ll'\{U-2x_0(U+W'+V')-Kx_0^2\}$$
$$+\Sigma(mn'+m'n)\{U'-z_0(W'+U'+V)-y_0(V'+U'+W)-Ky_0z_0\}=0.$$

Taking (x_0, y_0, z_0), so that

$$\frac{U-2x_0(U+W'+V')-Kx_0^2}{a^2}=\text{etc.}, \quad \ldots\ldots\ldots\ldots(1)$$

the condition of conjugacy becomes

$$\Sigma a^2 ll' - \Sigma bc \cos A(m'n+m'n)=0,$$

which is also the condition for perpendicularity.

But if (x_0, y_0, z_0) be such a point that all lines through it are at once conjugate for the conic and perpendicular to each other, then (x_0, y_0, z_0) is a focus. Hence the equations (1) are the equations for the foci.

If $(\bar{x}, \bar{y}, \bar{z})$ be the centre, $-K\bar{x}=U+W'+V'$ (by Sect. II. 5);

$$\therefore U-2x_0(U+W'+V')-Kx_0^2=U+K\{\bar{x}^2-(x_0-x)^2\},$$

and

$$U+K\bar{x}^2=-\frac{H}{K}(v+w-2u')=\frac{H}{K}\frac{dK}{du};$$

$\therefore$ putting $x_0-\bar{x}=\xi_0$, we have for the foci the system of equations

$$\left\{K\xi_0^2-(H/K)\frac{dK}{du}\right\}\Big/a^2=\text{etc.}$$

Hence there are four foci, real or imaginary, lying on the axes symmetrically with regard to the centre.

Corollary. If the tangents from (x_0, y_0, z_0) pass through the circular points at infinity, the equation of these tangents must satisfy the conditions for a circle, viz.,

$$\left\{4\frac{dK}{du}\cdot\phi(x_0, y_0, z_0)+(\phi_{y_0}-\phi_{z_0})^2\right\}\Big/a^2=\ldots=\ldots,$$

or, after some reduction,

$$\{U-2x_0(U+V'+W')-Kx_0^2\}/a^2=\ldots=\ldots.$$

Hence, by above, the equation to the tangents from a focus satisfies the conditions for a circle, and the conic is inscribed to a quadrilateral whose vertices are the foci and the circular points at infinity.

The foci of the conic $\Sigma\sqrt{lx}=0$ are given by the equations

$$\frac{lx^2-x(y+z)(m+n)}{a^2}=\ldots=\ldots.$$

8. *Special forms of the equations of a pair of conics.*

In general two conics intersect in four distinct points (real or imaginary). Take the three points of intersection of the pairs of lines joining these points for the vertices of the triangle of reference; then the polar of each vertex of the triangle with respect to either conic is the opposite side (for proof see any text-book on Pure or Analytical Geometry), and thus the equations of the conics may be written in the form $\Sigma lx^2=0$, $\Sigma Lx^2=0$. It does not necessarily happen that the triangle of reference is real, but the *analytical* result is independent of this.

If the two conics touch one another, the vertices of the triangle taken in this manner lie in a line. In this case, by taking the common tangent for a side of the triangle, its point of contact for a vertex, and suitably choosing the remaining sides, we may put the equations in the following forms:

when the conics touch,

$$\left.\begin{aligned} ux^2+2v'zx+\ 2fyz=0\\ ux^2+2v'zx+2Fyz=0\end{aligned}\right\};$$

when the conics have double contact,

$$\left.\begin{aligned} ux^2+\ 2fyz=0\\ ux^2+2Fyz=0\end{aligned}\right\};$$

when the conics have contact of the second order,

$$\left.\begin{aligned} ux^2+2u'yz+\ 2fzx=0\\ ux^2+2u'yz+2Fzx=0\end{aligned}\right\};$$

when the conics have contact of the third order,

$$\left.\begin{aligned} ux^2+2v'yz+\ fz^2=0\\ ux^2+2v'yz+Fz^2=0\end{aligned}\right\}.$$ [*Trin.* (*Cam.*), 1899.]

Section IV.

The Circle.

Applying the conditions, Sect. III. 2, the most general equation of a circle is

$$\Sigma ux\Sigma x+\Sigma a^2yz=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Hence *all circles intersect the line at infinity in the same two points*, viz., the intersections of $\Sigma x=0$, $\Sigma a^2yz=0$.

The radius of (1) is $\sqrt{H/K}$.

Aliter. Let the curve represented by the general equation of the second degree cut the sides of the triangle of reference in P_1, P_2; Q_1, Q_2; R_1, R_2 respectively.

If the curve be a circle, we have $AR_1 . AR_2=AQ_1 . AQ_2$, etc. Let p, q, r be the lengths of the tangents to the circle from A, B, C respectively.

Now at R_1, R_2, $ux^2+vy^2+2w'xy=0$ and $x+y=1$. Solving for these, we have for root relations $x_1x_2/v=y_1y_2/u=1/(u+v-2w')$,

But $AR_1 . AR_2=c^2y_1y_2$; therefore $p^2(u+v-2w')=c^2u$, whence

$$(u+v-2w')/c^2=u/p^2=\text{by symmetry } (u+w-2v')/b^2.$$

Hence the conditions for a circle are

$$(v+w-2u')/a^2=\ldots=\ldots=u/p^2=v/q^2=w/r^2,$$

and the general equation may be written $\Sigma x \Sigma p^2 x - \Sigma a^2 yz = 0$, where p, q, r are the powers of the vertices of the triangle of reference.

The following are derived at once :

Circumcircle,

$$\Sigma a^2 yz = 0. \quad \ldots\ldots (2)$$

In-circle,

$$\Sigma a^2 yz - \Sigma x\overline{s-a}^2 \Sigma x = 0, \qquad \text{or } \Sigma \sqrt{x(s-a)} = 0. \quad \ldots\ldots (3)$$

Ex-circle,

$$\Sigma a^2 yz - (\Sigma x)[s^2 x + \overline{s-c}^2 y + \overline{s-b}^2 z] = 0, \text{ or } \sqrt{sx} + \sqrt{\overline{s-c}\,y} + \sqrt{\overline{s-b}\,z} = 0. \quad \ldots (4)$$

Nine-point circle,

$$\Sigma a^2 yz - \tfrac{1}{4}\Sigma x \Sigma x(b^2 + c^2 - a^2) = 0. \quad \ldots\ldots (5)$$

Radical Axis.

If $S_1 \equiv \Sigma a^2 yz - \Sigma p_1^2 x \Sigma x = 0$ and $S_2 \equiv \Sigma a^2 yz - \Sigma p_2^2 x \Sigma x = 0$ be the equations to two circles, the equation of the radical axis is $\Sigma x(p_1^2 - p_2^2) = 0$.

Hence we see at once that the radical axes of three circles taken in pairs are concurrent.

Feuerbach's Theorem.

The radical axes of the N.P. and the in- and ex-circles are respectively

$$\Sigma x/(b-c) = 0, \qquad x/(b-c) + y/(c+a) - z/(a+b) = 0,$$

$$-x/(b+c) + y/(c-a) + z/(a+b) = 0, \quad x/(b+c) - y/(c+a) + z/(a-b) = 0.$$

These lines touch respectively the in- and ex-circles ;

$\therefore$ the N.P. circle touches the in- and ex-circles.

Length of Tangent.

If ρ_1, ρ_2 be the distances along $\Sigma lx = 0$ from (x_0, y_0, z_0) to the circle,

$$\psi(x, y, z) \equiv \Sigma x \Sigma p^2 x - \Sigma a^2 yz,$$

then $$\rho_1 \cdot \rho_2 = t^2 \psi(x_0, y_0, z_0)/\psi(m-n, n-l, l-m).$$

But $$\psi(m-n, n-l, l-m) = -\Sigma a^2 \overline{n-l}\;\overline{l-m} = t^2 ;$$

$$\therefore \ \rho_1 \cdot \rho_2 = \psi(x_0, y_0, z_0) ;$$

$\therefore$ square of tangent from (x_0, y_0, z_0) to circle is $\Sigma x_0 \Sigma p^2 x_0 - \Sigma a^2 y_0 z_0$.

Example. All circles touching $\Sigma yz = 0$ at $y = 0, z = 0$ are given by the equation

$$(q^2 y + r^2 z)(\Sigma x) - 2\Sigma a^2 yz = 0, \text{ where } q^2 - c^2 = r^2 - b^2 = \lambda, \text{ say.}$$

This equation may be written $(y+z)(q^2 y + r^2 z) - (a^2 + \lambda) yz + \lambda \Sigma yz = 0$, and so the common chords of this circle and $\Sigma yz = 0$ through $y = 0, z = 0$ are given by $(y+z)(q^2 y + r^2 z) - (a^2 + \lambda) yz = 0$. If the circle be the osculating circle then $(y+z) = 0$ must be one of these common chords, so $\lambda = -a^2$, and the other common chord is $y(a^2 - c^2) + z(a^2 - b^2) = 0$. [*Corpus* (*Cam.*), 1899.]

Example.—The equations of the escribed circles of the triangle of reference being $[s^2 x + (s-c)^2 y + (s-b)^2 z]\Sigma x - \Sigma a^2 yz = 0$, etc., the radical axes of these circles meet in the point $\frac{x}{b+c} = \ldots = \ldots = \frac{1}{4s}$. The circle orthogonal to the escribed circles has this point for centre, and the square of its radius is obtained by substituting these values of x, y, z in the equation of any one of the escribed circles; hence $(\text{radius})^2 = \frac{1}{4}\left(r^2 + \frac{\Delta^2}{r^2}\right)$, where r is the radius of the in-circle. The square of the tangent from the vertex A to this circle is therefore $-(s-b)(s-c)$, and thus its equation is $\Sigma x \Sigma x(s-b)(s-c) + \Sigma a^2 yz = 0$.

Section V.

The Parabola.

The General Equation of the Second Degree represents a Parabola if $K=0$ (§ III. 4). From this it follows that:

(1) The centre is at infinity; (2) the only focus in the finite plane is that given by $[U-2x_0(U+W'+V')]/a^2=\ldots=\ldots$; (3) the lengths of the axes are infinite. The (Semi-Latus Rectum)^2 is

$$16H\Delta^4\Big/\Big[\Sigma bc\cos A\frac{dK}{du}\Big].$$

The axes of the general conic (III. 3) are

$$\sum\frac{(x-\bar{x})^2}{a^2}\Big(\frac{1}{b^2}\frac{dK}{dv}-\frac{1}{c^2}\frac{dK}{dw}\Big)=0,$$

where $\bar{x}=-(U+W'+V')/K$, etc. Hence

$$\bar{x}^2=-U/K+(H/K^2)\frac{dK}{du},\text{ etc.}\ldots,$$

and the equation to axes becomes

$$\sum\frac{x^2}{a^2}\Big(\frac{1}{b^2}\frac{dK}{dv}-\frac{1}{c^2}\frac{dK}{dw}\Big)+\sum\frac{1}{K}\Big[\frac{2x(U+W'+V')-U}{a^2}\Big(\frac{1}{b^2}\frac{dK}{dv}-\frac{1}{c^2}\frac{dK}{dw}\Big)\Big]=0.$$

Thus, as $K=0$, one axis is at infinity, and the other is given by

$$\sum\frac{2x(U+W'+V')-U}{a^2}\Big(\frac{1}{b^2}\frac{dK}{dv}-\frac{1}{c^2}\frac{dK}{dw}\Big)=0,$$

which clearly passes through the focus.

The equation to the Directrix is obtained from that of the Director Circle of the general conic and is

$$\Sigma x(b^2W+c^2V+2bcU'.\cos A)=0.$$

Hence, as the pole of the Directrix is the focus, we have

$$\frac{ux_0+v'y_0+w'z_0}{b^2W+c^2V+2bcU'.\cos A}=\ldots=\ldots,$$

which is transformable into the result already obtained.

Finally, if the equation be transformed to the simple form $\Sigma lx^2=0$, it represents a parabola if $\Sigma mn=0$.

For focus:

$$\frac{1-2x_0}{a^2l}=\ldots=\ldots,\text{ or }\frac{x_0}{b^2m+c^2n}=\ldots=\ldots,$$

$$(\tfrac{1}{2}\text{ Lat. Rect.})^2=-16\Delta^4lmn/\Sigma a^2l,$$

Equation of Directrix is $\Sigma lx(b^2m+c^2n)=0$,

Equation of Axis is $\Sigma\cos A(my-nz)/(al)=0$.

This passes through the point $lx=my=nz$, which lies on the curve and on $\Sigma x=0$.

$\therefore$ $lx=my=nz$ is the point of contact of the parabola and the line an Infinity.

REVIEW.

Analyse Infinitésimale, à l'usage des Ingénieurs. Par MM. ROUCHÉ ET LÉVY. Tome premier, calcul différentiel, pp. 557. Gauthier-Villars, Paris, 1900 (*Encyclopédie Industrielle*).

ON taking up this volume we are at once led to contrast the range of studies required from the French *Ingénieur* with that of the English *Engineer*. A slight inspection of the mathematical papers set in the Cambridge Mechanical Sciences Tripos would shew that the bulk of the work under review falls outside the Cambridge curriculum; and yet we have heard that some of the older school of English engineers regard the Cambridge course as too full of theory. Possibly the mental difference of the French and English may be partly responsible for this; the English seem inclined to be content with "rule of thumb" methods, while the attitude of the French may be summed up in the last words of the preface of MM. Rouché et Lévy: "le lecteur français ne se complait guère aux assertions dont il doit chercher ailleurs la preuve."

We think that this book will be found an excellent introduction for the English reader to the more exact treatises of Jordan and Harnack; MM. Rouché et Lévy employ a judicious seasoning of modern views on analysis without using so much of them as to weary a student who is unfamiliar with "arithmetized" mathematics. The work should prove an interesting guide towards that precision of analysis which is so marked in Continental works, and so sadly deficient in many of the standard English text-books. The treatment of infinite series deserves careful study, though the first chapter (9) on this subject contains little that will be novel to readers of Chrystal's *Algebra*, or Hobson's *Trigonometry*; however, Kummer's and Gauss's tests for convergence are more fully considered by the French authors. The second chapter (14) is, we believe, entirely untouched by English writers of elementary text-books, with the exception of Harkness and Morley. We think that the chapters containing geometrical applications (16-19) are probably handled more scientifically than in most English books; particularly the careful examination of a plane curve near a singular point by means of "Newton's parallelogram" (Arts. 313-316). MM. Rouché et Lévy give a more careful discussion than Edwards (in his *Differential Calculus*), while it seems (at any rate to the present reviewer) more attractive than Chrystal's or Harnack's; the defect in Edwards's investigation is that it leaves out of account the possibility that more than two representative points may be on the same side of the polygon in Newton's diagram. Owing to the misleading use of the term "double cusp," applied by Edwards to a point of contact of two distinct branches of a curve, we should advise readers to study carefully the explanation and figures (pp. 316, 317) of the possible varieties and true nature of this singularity.

However, we think it desirable to point out that many English readers will find it easy to lighten the analysis in places (without loss of rigour) by the use of a few well-known Cambridge "tips." For instance, Arts. 371, 372 can be made simpler by the use of the familiar expressions* for the coordinates of points of a curve, referred to the tangent and normal at some point of the curve,

$$x = s - \tfrac{1}{6}\kappa^2 s^3 - \tfrac{1}{8}\kappa\kappa' s^4 - \ldots, \quad y = \tfrac{1}{2}\kappa s^2 + \tfrac{1}{6}\kappa' s^3 + \tfrac{1}{24}(\kappa'' - \kappa^3) s^4 + \ldots,$$

where κ = curvature at the origin, s = arc measured from the origin. It may be remarked incidentally that the second result in Art. 372 should be $\frac{1}{6}(x'/x)(ds)^3$, instead of $\frac{1}{10}(\kappa'/\kappa)ds$. Of course the same suggestion can be applied to the three dimensional cases (*e.g.* Arts. 400, 416); indeed the authors make use of it in Art. 418. The methods of "moving axes" in connection with solid geometry solve many of the problems which are here treated by Frenet's formulae. For example, if a curve has the ratio of the curvature to the torsion equal to a constant ($\tan\alpha$) it is found that a line fixed in space can make angles α, $\frac{1}{2}\pi$, $\frac{1}{2}\pi - \alpha$ with the tangent, principal normal, binormal respectively and the results of Art. 423 follow at once; and similar methods could be used in Arts. 428, 429. Incidentally, the neatest form for the condition of Art. 427 is

$$dl.dl' + dm.dm' + dn.dn' = 0,$$

* See Wolstenholme's *Problems* (1770), or Edwards's *Differential Calculus*. The easiest investigation is by means of a recurrence formula between the coefficients of the powers of s.

where l,m,n,l',m',n' are the six Plücker coordinates of a line (all considered as functions of a parameter t).

The treatment of asymptotes might be improved as a whole by following Frost's method in his *Curve Tracing*; but Art. 306 is instructive. MM. Rouché et Lévy have called attention to two or three cases of exception to general results which, though commonplace on the Continent, may be sufficiently novel here to be worth mentioning. Thus (Art. 34) the functions $x\sin(1/x)$ and $x/(1+e^{1/x})$ have no determinate differential coefficient for $x=0$; an asymptote is not necessarily the limit of a tangent (Art. 303), as we see from $y=(\sin x)/x$, for which $y=0$ is an asymptote, but the limit of the tangent is not definite. The exceptional case of continuous functions without differential coefficients might have been mentioned; possibly the analytical details are beyond the range laid down by the authors, but simple geometrical figures (such as are given by Prof. E. H. Moore "On crinkly curves," *Trans. Am. Math. Soc.*, vol. i., 1900) could be made clear to almost any intelligent student on a second reading of the subject.

We should like to call attention to a not uncommon blunder which is not specifically pointed out by the French authors (possibly they would regard a warning as superfluous); this is the assumption that in an expression depending on two limits, the order of taking the two limits is immaterial; the statement is no doubt true "in general," but, as Prof. Bôcher points out in a recent paper, a theorem which holds "in general" may fail in the *special* case to which we wish to apply it. Readers of Art. 166 of Edwards's *Differential Calculus* will perhaps raise the objection that a proof is given there which covers all cases; it *appears* to do so, but an examination of the two limits

$$\operatorname*{Lt}_{x=0}\left(\operatorname*{Lt}_{y=0}\frac{x-y}{x+y}\right)=+1,\quad \operatorname*{Lt}_{y=0}\left(\operatorname*{Lt}_{x=0}\frac{x-y}{x+y}\right)=-1,$$

will shew that this is not the case. Of course it may appear that this exception is somewhat trivial; but similar exceptions may occur in differentiating or integrating an integral, taking the sum of a doubly infinite series, taking the value (for a particular value of x) of an infinite series of functions of x, etc. Thus consider a series whose sum to n terms is $\dfrac{1-nx}{1+nx}$; the value of this for $x=0$ is $(+1)$; but if we make n infinite first, we get the value (-1).

It is worth while, perhaps, to remark that the list of *Errata* given on p. vii. is by no means exhaustive; those which we have noticed hardly call for special remark, as the reader cannot fail to see the correct form. On the whole, the printing is not on the high level of the mathematical series by which we know Gauthier-Villars; of course the volume is not intended primarily for mathematicians, but is a part of the *Encyclopédie Industrielle*, a series of engineering text-books.

We have not attempted to criticise the work from the point of view of the public to whom it is addressed, because we do not think that the English School of Engineers could be induced to read it; but if we can persuade a few mathematicians, as yet unacquainted with continental analysis, to study the book, we think it may help to correct some impressions which are too common in this country. We wonder if an English book on the Calculus "for the use of engineers" has ever been suggested to French mathematical students as filling a gap in the French text-books!

T. J. I'A. BROMWICH.

ST. JOHN'S COLLEGE,
CAMBRIDGE, *5th April*, 1901.

MATHEMATICAL NOTES.

97. [L¹. 6. c.] *Trilinear equation of a circle of curvature.*

The general equation of a conic circumscribing the triangle of reference is $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, and the equation of the tangent at A is $m\gamma+n\beta=0$.

Hence the general equation of a conic having contact of the second order at A with the above conic is

$$l\beta\gamma+m\gamma\alpha+n\alpha\beta+(m\gamma+n\beta)(D\beta+E\gamma)=0,$$

and this will moreover represent the circle of curvature at A if it can be thrown into the form

$$P(a\beta\gamma+b\gamma a+ca\beta)+(aa+b\beta+c\gamma)\left(\frac{Dn}{b}\beta+\frac{Em}{c}\gamma\right)=0.$$

Equating coefficients,

$$l+Dm+En=Pa+\frac{Dcn}{b}+\frac{Ebm}{c},$$

$$m=Pb+\frac{Eam}{c},$$

$$n=Pc+\frac{Dan}{b}.$$

These equations give

$$P=almn/\{bc(m^2+n^2-2mn\cos A)\},$$
$$D=b/a-lm/\{m^2+n^2-2mn\cos A\},$$
$$E=c/a-ln/\{\ldots\ldots\ldots\ldots\ldots\ldots\}.$$

Now let us take $al=bm=cn=1$, so that the conic circumscribing the triangle of reference is the minimum ellipse whose centre is at G and tangents at A, B, C parallel to BC, CA, AB respectively.

Then

$$m^2+n^2-2mn\cos A=a^2/b^2c^2,$$

$$D=\frac{b}{a}-\frac{bc^2}{a^3}=\frac{b}{a^3}(a^2-c^2),$$

$$E=\frac{c}{a}-\frac{b^2c}{a^3}=\frac{c}{a^3}(a^2-b^2).$$

Hence the common chord of the ellipse and circle of curvature at A is

$$b\beta(a^2-c^2)+c\gamma(a^2-b^2)=0,$$

or

$$b\beta(c^2-a^2)=c\gamma(a^2-b^2).$$

The circles of curvature at A, B, C therefore cointersect in the point whose trilinear coordinates are inversely proportional to

$$a(b^2-c^2),\quad b(c^2-a^2),\quad c(a^2-b^2),$$

the STEINER point of ABC.

This point is the fourth point of intersection of the ellipse with the circumscribing circle of the triangle of reference, which may be seen by solving simultaneously

$$\frac{1}{aa}+\frac{1}{b\beta}+\frac{1}{c\gamma}=0,$$

$$\frac{a}{a}+\frac{b}{\beta}+\frac{c}{\gamma}=0,$$

or from the fact that

$$D\beta+E\gamma=0 \quad\text{and}\quad \frac{Dn}{b}\beta+\frac{Em\gamma}{c}=0$$

coincide if

$$\frac{n}{b}=\frac{m}{c} \quad\text{or}\quad m/n=(1/b)/(1/c).$$

R. F. DAVIS.

98. [L¹. 11.] *Condition for rectangular hyperbola in trilinears.*

If $ua^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma a+2w'a\beta=0$ represent a rectangular hyperbola, it is possible to describe about the triangle of reference another rectangular hyperbola $P/a+Q/\beta+R/\gamma=0$ (where $\Sigma(P\cos A)=0$, as it passes through the orthocentre), so that the asymptotes of the two curves are parallel.

It must be therefore possible to throw the given equation into the form

$$\Sigma(a \sin A) \cdot \Sigma(ua \operatorname{cosec} A) + \lambda \cdot \Sigma(P\beta\gamma) = 0.$$

Hence $v \sin C \operatorname{cosec} B + w \sin B \operatorname{cosec} C + \lambda P = 2u'$

.....

.....

and $u + v + w - 2u' \cos A - 2v' \cos B - 2w' \cos C = 0.$

R. F. DAVIS.

99. [P. 2. a.] *Extension of Problem* 394.

Problem 394 will be found true for a curve S of any order, the theorem on which it depends being the following:

Let κ be the curvature of a curve S at a point P, the radius vector from the origin O to which is r. Corresponding to the tangent at P is a point P' on the reciprocal curve Σ (taken with respect to centre O, radius k). Let κ', r' be the curvature and radius vector at P'. Then $\kappa r^3 = \frac{k^4}{\kappa' r'^3}$ so that the quantity κr^3 is inverted by reciprocation.

The following corollaries are obvious:

If S has a cusp at P, Σ has an inflexion at P'.

If S has a tacnode at P, the curvatures of the branches being κ_1, κ_2, Σ has a tacnode at P', the curvatures being κ_1', κ_2', where $\kappa_1 : \kappa_2 = \frac{1}{\kappa_1'} : \frac{1}{\kappa_2'}$.

We may also obtain from this theorem an expression for the curvature of a conic S at any point P. Take the origin O at a focus of the conic S. Let P be a point on the conic, $OP = r$. Reciprocate S with respect to origin O, radius r. The reciprocal conic Σ is a circle of radius $\frac{r^2}{l}$, where l is semi latus-rectum of S. Since the circle whose centre is O, radius r, reciprocates into itself, the tangent at P to this circle will touch the circle Σ at a point P'. Let $OP' = r'$, then the theorem $\kappa r^3 = \frac{k^4}{\kappa' r'^3}$ gives $\kappa = \frac{1}{l}\left(\frac{r}{r'}\right)^3$ or radius of curvature at $P = l \sec^3\theta$ where $\theta = P\hat{O}P'$.

This expression is easily reducible to the normal form $\kappa = \frac{ab}{CD^3}$.

A. P. THOMPSON.

100. [K. 1. b.] The following proofs of the "Equal Bisectors Theorem" are perhaps worthy of record, if they have not already been published. Mr. Davis's Proof in the December Number, 89 [K. 1. b.], employs unnecessarily "heavy metal" (Ptolemy's Theorem, etc.).

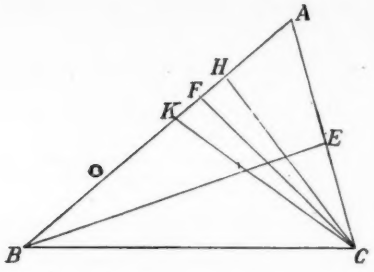
The theorem is an obvious consequence of this:

If in the triangle ABC the side AB is greater than AC, the bisector BE of the angle B terminated in AC is greater than the bisector CF of the angle C terminated in AB.

First Proof. Make the angles ACH, BCK each equal to $\frac{1}{2}B$,
then since $C > B$, CH, CK will lie on opposite sides of CF.

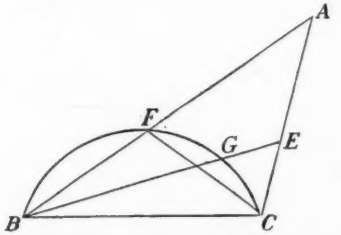
The triangles ACH, ABE are equiangular and $AB > AC$,
therefore $BE > CH$.

Also in the triangles BCK, BCE,
BC is common, $BCK = CBE$ and $KBC > BCE$,
therefore $BE > CK$.



But CF, lying between CK, CH, must be less than the greater, so that, being less than (or equal to) one of them which is less than BE,

CF *is* less than BE. Q.E.D.



Second Proof. Describe the circular arc BFC, cutting the bisector of B in G.

Then $BFC = A + \frac{C}{2}$, $BEC = A + \frac{B}{2}$, so that $BFC > BEC$, and $\therefore$ $BGC > BEC$, hence G lies between B and E.

Also since BCF or $\frac{1}{2}C > EBC$ or $\frac{1}{2}B$, arc $BF >$ arc CG,

hence arc $BG >$ arc CF and chd. $BG >$ chd. CF.*

$\therefore$ *à fortiori*, $BE > CF$. Q.E.D.

This proof is due to the Rev. H. W. Watson, D.Sc., F.R.S., somewhat improved by the late H. R. Droop. R. B. Hayward.

PROBLEMS.

[*Much time and trouble will be saved the Editor if (even tentative) solutions are sent with problems by their Proposers.*]

430. [L¹. 10. d.] Given two straight lines of fixed lengths and placed perpendicular to each other, what must be their relative positions in order that a real parabola may pass through their extremities? E. N. Barisien.

431. [K. 1. b. α.] If I be the incentre of a triangle ABC and BE, CF the external bisectors terminated by the opposite sides, then if IB, IA, IC are in geometrical progression, $BE = CF$. R. B. Hayward.

432. [A. 3. α.] The discriminant Δ of a polynomial $f(x)$ being expressed in the form $\phi(x)f(x) + \chi(x)f'(x)$, prove that when $\Delta = 0$ the common factor of $f(x)$ and $f'(x)$ is also a factor of $\chi(x)$ and $\phi(x) + 2\chi'(x)$.

R. W. H. T. Hudson.

* This follows only if BF, BG are on the same side of the diameter through B of the circle BFC. $\left[BF = \frac{BC}{BC+CA} \cdot BA < BC;\ \therefore$ diameter does not lie between BF and $BG\right]$.

433. [K. 13. c.] a, b, c, d are the sides taken in order and e, f the diagonals of a skew quadrilateral; shew that

$$\cos\hat{bc}\cos\hat{ad}-\cos\hat{ca}\cos\hat{bd}+\cos\hat{ab}\cos\hat{cd}=(c^2a^2+b^2d^2-e^2f^2)/2abcd.$$

To what does this reduce in plano? [Cf. 387.] C. E. M'VICKER.

434. [K. 21. β.] Divide a given finite straight line into n equal parts by the use of a pair of compasses only. E. M. RADFORD.

435. [L¹. 10. a.] If PM be the perpendicular from the point P of a parabola on the tangent AM at the vertex A, find the locus of the foot of the perpendicular from M on AP. V. RETALI.

436. [L¹. 3. a.; 17. e.] The triangle of reference being the diagonal triangle of a given quadrilateral, shew that, if the locus of centres of conics inscribed to the quadrilateral is a diameter of the circumcircle of the triangle, then the conic $\Sigma a^2 \tan A=0$ is inscribed to the quadrilateral. A. S. TOLLER.

437. [K. 3. d.] The point P is the pole with respect to the Brocard circle of the chord $\Omega\Omega'$ [*i.e.* it is the point (a^3, b^3, c^3)]; find the equation of the circle $P\Omega\Omega'$, and if ρ', D are respectively the radius of the Brocard circle and the diameter of $P\Omega\Omega'$, prove that $D+\rho'=(D-\rho')\cot^2\omega$. R. TUCKER.

438. [R. 9. b.] A rigid square $ABCD$ of four smooth wires is placed with AC vertical, and two small equal spherical beads, whose coefficient of restitution is e, start from B and D simultaneously to slide down BC, DC; shew that their velocities of approach and separation at D are in the ratio $1:e$, and that after impact they separate till their distance apart is e^2BC. C.

439. [R. 4.] A small ring of weight w on a smooth elliptical wire (axis vertical) is tied to the foci by two elastic strings, the modulus of elasticity for each being $w/2$, and each is just unstretched when the ring is nearest to the corresponding focus; find the positions of equilibrium and determine in each case the nature of the equilibrium. C.

SOLUTIONS.

UNSOLVED QUESTIONS.—171, 275, 279, 283, 326, 336-8, 341, 349, 356, 370, 373-9, 381-2, 389, 393, 395, 397, 401, 404, 406, 414.

Solutions to the above, or other questions to which no solution has yet been published, and to 430-39 should be sent as early as possible.

The question need not be re-written; the number should precede the solution. Figures should be **very carefully drawn to a small scale** on a separate sheet.

60. [K. 12. b.] *To inscribe three mutually tangent circles in an isosceles triangle.* PROF. D. S. WRIGHT.

Solution.

Bisect the base AC in D; produce DB indefinitely.

Bisect $\hat{BCD}$ and $\hat{BDC}$ by lines meeting in O.

Draw OP parallel to CA cutting BD in E, and make $EP=OE$.

O and P are centres of two of the inscribed circles.

Draw a parallel to CB at a distance equal to OE, cutting DB produced in F. Join FO, and draw OG perpendicular to the parallel. Let GO meet FD in H.

Describe, with centre H and radius HG, a circle cutting FO produced in K.

Join HK and draw OQ parallel to HK and meeting BD in Q.

Q is the centre of the third circle.

Note.—The general case is known as Malfatti's problem. Steiner's construction was:—I being the incentre, inscribe circles $A'B'C'$ to the triangles BIC, CIA, AIB. Let SS', TT', UU' be the common tangents. Then the inscribed circles of ATT', BUU', CSS' are the circles required.

81. [A. 1. c. β.] If $x^3=234\sqrt{2}+148\sqrt{5}$ *and* x *is real, prove that* $x+\frac{2}{x}=4\sqrt{5}$.

Solution.

If $x+\frac{2}{x}=4\sqrt{5}$, $x^2+\frac{4}{x^2}-2=74$;

$$\therefore\ x^3+\frac{8}{x^3}=296\sqrt{5};$$

$$\therefore\ x^6-296\sqrt{5}\,x^3+8=0;$$

and
$$x^3=148\sqrt{5}\pm\tfrac{1}{2}\sqrt{296^2\,.\,5-32}$$
$$=148\sqrt{5}\pm234\sqrt{2};\quad\therefore\ \text{etc.}$$

82. [A. 1. a.] *Prove that* $\Pi(a-bc)+\Pi(1-a^2)$ *is divisible by* $(1-abc)$, *and find the quotient.*

Solution.

The quotient is $1+2abc-\Sigma a^2$.

83. [A. 1. a.] If a, b, c *are in* A.P. *and* x, y, z *are in* G.P., *prove that*

$$x^by^cz^a=x^cy^az^b.$$

Solution.

Writing $y=rx$, $z=r^2x$, and noting $c-b=b-a=\frac{1}{2}(c-a)$, the result required is true if $x^{c-b}\,.\,(rx)^{a-c}(r^2x)^{b-a}=1$, an identity.

87. [L¹. 10. a.] *If a tangent be drawn to a parabola at* Q, *and from any point* O *on the tangent,* OPD *be drawn parallel to the axis meeting the curve in* P, *and* QD *be perpendicular to* OP, *prove that* $4AS\,.\,OP=QD^2$. T. Roach.

Solution.

Draw QV parallel to the tangent at P. Then $OP=PV$ and $QV^2=4SP\,.\,PV$. Also if SY be perpendicular from S on the tangent at P, then $\triangle$s QDV, SYP are similar,

$$\therefore\ \frac{QD^2}{QV^2}=\frac{SY^2}{SP^2}=\frac{SA}{SP}$$

$$\therefore\ QD^2=\frac{SA}{SP}\,.\,4SP\,.\,PV=4SA\,.\,OP.$$

92. [K. 8. b.] *ABCD is a cyclic quadrilateral. Shew that the incentres of the triangles ABC, ADC, ABD, CBD are at the corners of a rectangle.*

Trin. (C.), '95.

Solution.

Let E, F, G, H be the mid points of the arcs AB....

Then the sum of the arcs HG, EF is half the circumference.

If EG, FH meet in I, we have $\angle EIF=\angle HEI+\angle IHE=\angle$ on arc $(HG+EF)=90°$.

Now J, K are on the circle centre E, radius EA, $\therefore JK$ the base of the isosceles triangle EJK is perpendicular to the bisector EG of the angle DEC, $\therefore$ etc.

Note i.—If O be the circumcentre; H_a, H_b, H_c, H_d the orthocentres of the triangles BCD, etc.; G_a, G_b, etc. their centres of gravity. Then, since the perpendicular from O on BD is half of H_aC and from O on AD is half of H_bC, we have H_bH_a is equal and parallel to AB.

Whence it is readily seen that $H_aH_bH_cH_d$ is a cyclic quadrilateral and equal to $ABCD$.

Note ii.—The Simson line of A with reference to BCD bisects AH_a.

Also AH_bH_aB is a parallelogram.

Hence AH_a, H_bB intersect at their middle points.

$\therefore$ AH_a, BH_b, CH_c, DH_d intersect at their common middle point, and this is the point of concurrence of the Simson lines of A, B, C, D with reference to the triangles BCD, etc.

Note iii.—We find $\frac{MG_a}{MB}=\frac{1}{3}=\frac{MG_b}{MA}$ where M is mid point of DC;

$\therefore$ G_bG_a is parallel to AB and one third of it;

$\therefore$ $G_aG_bG_cG_d$ is a cyclic quadrilateral.

93. [K. 8. b.] *The tangents at the extremities of two perpendicular chords of a circle form a cyclic quadrilateral.* Gonv. and Caius, '95.

Solution.

Let DFA, AEB, BHC, CGD be the tangents at the extremities of perpendicular chords FH, EG.

$$EBH=\pi-2BEH=\pi-2EGH;$$

$$FDG=\pi-2DFG=\pi-2FEG=\pi-2\left(\frac{\pi}{2}-EFH\right)=2EGH.$$

$\therefore$ $B+D=\pi$. $\therefore$ etc.

95. [A.1.a] *Resolve into factors*

$$(bcd+cda+dab+abc)^2-abcd(a+b+c+d)^2.$$ Queens' (Cam.), '96.

Solution.

Expn. $=(cd-ab)(bd-ac)(bc-ad)$.

96. [A. 2. b.] *Solve the equation* $ax^2+2bx+c+\lambda(x^2+1)=0$. *Find between what limits of* λ *it has equal roots, and show that no positive value of* λ *will make the roots real unless the roots of* $ax^2+2bx+c=0$ *are real.*

Clare and Trin. Hall, '96.

Solution.

Taking the more general case

$$\lambda(ax^2+bx+c)+\mu(a'x^2+b'x+c')=0$$ Magd. (C)., '91

the condition for real roots is that

$$(\lambda b+\mu b')^2-4(\lambda a+\mu a')(\lambda c+\mu c')>0$$

or $$\lambda^2(b^2-4ac)+2\lambda\mu\{bb'-2(a'c+ac')\}+\mu^2(b'^2-4a'c')>0,$$

which requires, for real values of λ/μ, $b^2-4ac>0$

or $$(ac'-a'c)^2-(bc'-b'c)(ab'-a'b)>0.$$

To find values for equal roots change $>$ into $=$.

97. [A. 1. a. I. 1.] *If* Q *be the quotient and* R *the remainder when a multiple of* 67 *is divided by* 10, *then* $Q-20R$ *is a multiple of* 67.

Clare and Trin. Hall, '96.

Solution.

Let $$67k=10Q+R$$

and $$p=Q-20R.$$

Then $$20\,.\,67k+p=3\,.\,67\,.\,Q.$$

$\therefore$ p is a multiple of 67.

98. [J. 2. f.] *If* n^2 *coins, of which exactly* n *are silver, are arranged at random in* n *rows, each containing* n *coins, the chance that one row at least occurs in which there is no silver coin is* $1-\frac{(n-1)!(n^2-n)!\,n^{n-1}}{(n^2-1)!}$.

King's and Pem., '95.

Solution by W. A. Whitworth.

We are not concerned with the *order* of the coins in any row; it is simply a case of distribution into rows. n^2 coins can be distributed into n rows of n

each in $\frac{(n^2)!}{(n!)^n}$ ways; but if the silver coins are to be one in each row the number of ways is $n! \times \frac{(n^2-n)!}{\{(n-1)!\}^n}$.

The chance that there should be one silver coin in each row is therefore

$$\frac{n^{n-1}(n-1)!(n^2-n)!}{(n^2-1)!},$$

and the required chance is the complement of this.

99. [K. 20. c. a.] *Prove* $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8}$. Queens' (C.), '96.

v. Tod. *Trig.* (1893), p. 226.

100. [K. 20. d.] *If* $\cos\frac{\Sigma\alpha-\pi}{4}+\Sigma\cos\frac{3\alpha-\beta-\gamma+\pi}{4}=0$, *then*

$$\Pi\tan\frac{\beta+\gamma-\alpha}{4}=1.$$ Trin. (C.), '95.

Solution.

We have $2\cos\frac{\alpha}{2}\cos\frac{\beta+\gamma-\alpha-\pi}{4}+2\cos\frac{\beta+\gamma-\alpha+\pi}{4}\cos\frac{\beta-\gamma}{2}=0,$

or $$\cos\frac{\alpha}{2}\cos\frac{\beta+\gamma-\alpha}{4}+\cos\frac{\alpha}{2}\sin\frac{\beta+\gamma-\alpha}{4}$$
$$+\cos\frac{\beta+\gamma-\alpha}{4}\cos\frac{\beta-\gamma}{2}-\sin\frac{\beta+\gamma-\alpha}{4}\cos\frac{\beta-\gamma}{2}=0;$$

$$\therefore\ 2\cos\frac{\beta+\gamma-\alpha}{4}\cos\frac{\alpha+\beta-\gamma}{4}\cos\frac{\alpha-\beta+\gamma}{4}$$
$$-2\sin\frac{\beta+\gamma-\alpha}{4}\sin\frac{\alpha+\beta-\gamma}{4}\sin\frac{\alpha-\beta+\gamma}{4}=0;$$

$\therefore$ etc.

101. [K. 20. e. a.] *At a point within a rectangle the sides subtend angles* $\alpha, \beta, \gamma, \delta$ *in order. Prove that the ratio of the sides is given by*

$$x^2\sin\beta\sin\delta-x\sin(\beta+\delta)+\sin\alpha\sin\gamma=0.$$ Trin. (C.), '95.

Solution.

Let $AOB=\alpha$, etc. $AO=p$; $OB=q$; $OAB=\theta$; $AB=a$; $BC=b$.

Then $\frac{p}{a}=\frac{\sin(\alpha+\theta)}{\sin\alpha}$; $\frac{q}{a}=\frac{\sin\theta}{\sin\alpha}$; $\frac{p}{b}=\frac{\cos(\theta-\delta)}{\sin\delta}$; $\frac{q}{b}=\frac{-\cos(\alpha+\theta+\beta)}{\sin\beta}$;

$$\therefore\ \frac{a}{b}=-\frac{\sin\alpha}{\sin\beta}\frac{\cos(\theta+\alpha+\beta)}{\sin\theta}=\frac{\sin\alpha}{\sin\delta}\frac{\cos(\theta-\delta)}{\sin(\theta+\alpha)},$$

whence by reduction $x\sin\beta=-\sin\alpha\{\cot\theta\cos(\alpha+\beta)-\sin(\alpha+\beta)\}$,

$$x\sin\delta=\sin\alpha\left(\frac{\sin\delta+\cot\theta\cos\delta}{\cos\alpha+\cot\theta\sin\alpha}\right),$$

and eliminating θ, we obtain

$$x^2\sin\beta\sin\delta-x\sin(\beta+\delta)+\sin\alpha\sin\gamma=0.$$

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[*Sufficient Conditions in the Calculus of Variations.* Osgood. *Lagrange's Equations in the Calculus of Variations and the Extensions of a Theorem of Eldmann.* Whittemore. *On Some Points in the Theory of the Hypergeometric Function expressed as a Double Circuit Integral.* Hathaway. *A Theorem in Continued Fractions.* Lehmer. *Note on the Dual of a Focal Property of the Inscribed Ellipse.* Allardice. *A Simplified Solution of the Cubic.* M'Clintock.]

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* Will be reviewed shortly.

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